Theoretical Computer Science 405 (2008) 24–34

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Finite relational structure models of topological spaces and maps

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ARTICLE INFO

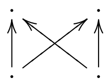
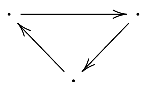
Keywords:
Binary reflexive relational structure
Weak homotopy equivalence
Homotopy pushout
Double mapping cylinder

ABSTRACT

We define the notion of homotopy pushout in the category of binary reflexive relational structures and explore its basic properties. We construct finite models in this category, of spaces and maps in **Top** with a view to developing systematic methods in this regard.

1. Introduction

A partial order on a set X determines a T_0 topology on X. This topological space TX has as a basis the collection $\{U_x : x \in X\}$ where, for each $x \in X$, $U_x = \{y \in X : y \leq x\}$. The assignment is functorial and in particular, T induces an isomorphism between the category **Fpos** (of finite partially ordered sets and order-preserving functions) and the category of finite T_0-spaces and continuous maps. There is also a link between finite posets and compact polyhedra via a functor $\mathcal{K}$ defined by Alexandroff [2]. For a finite poset X, $\mathcal{K}X$ is a simplicial complex whose vertices are the points of X, and whose simplexes are the totally ordered subsets of X. McCord [13] showed that TX has the same homology and homotopy groups as the polyhedron $|\mathcal{K}X|$ by constructing a natural weak homotopy equivalence $q_X : |\mathcal{K}X| \to TX$. Following this connection, examples of finite poset models of sphere multiplications and Hopf constructions were given in [8] and [7].

A partial order relation on a set X is reflexive, transitive and antisymmetric. The latter two conditions are not always conducive to forming quotient objects (i.e., when making identifications in the relevant set X). Hence, it would be more convenient to search for finite models of the form (X, θ), where θ is just a reflexive relation, instead of insisting on poset models. Moreover, it is possible to find finite $\mathcal{R}$-models of smaller cardinality if we are willing to sacrifice transitivity. For example, although the smallest poset model of a circle is the 4-point crown , we may contemplate instead the 3-point model , noting that the relevant relation fails to be transitive.

The category $\mathcal{R}$ of reflexive binary relational structures (X, θ) is known to be *topological* over the category **Set** of sets and functions [1, Definition 21.7 and Example 21.8(1)]. Although long recognised in categorical topology (e.g. see [14]), relational structures have been used recently by Larose and Tardif [10] to study *complexity class* problems. In particular, these authors have internalised in $\mathcal{R}$ the notions of homotopy group and weak homotopy equivalence.

It is fair to say that homotopy-theoretic notions have taken some time to develop in general topological categories such as in [10] or the work [4] of Grandis even though they began in **Top**. The purpose of the present paper is to show that the concept of *homotopy pushout* can also usefully be defined in $\mathcal{R}$. Indeed, our main result (Theorem 5.5) asserts that a model

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0304-3975/$ – see front matter
doi:10.1016/j.tcs.2008.06.021

in $\mathcal{R}$ of the double mapping cylinder $M(g_1, g_2)$ of a cotriad $A_1 \xleftarrow{g_1} A_0 \xrightarrow{g_2} A_2$ in **Top** can be constructed as the *double mapping fence-cylinder* $Z(f_1, f_2)$, where f_1 and f_2 are $\mathcal{R}$-models of g_1 and g_2 respectively.

Section 2 is devoted to defining the category $\mathcal{R}$, and briefly to discuss barycentric subdivision and geometric realizations. The construction of push-outs is relatively easy, and is presented in Section 3. In Section 4 we define homotopy pushouts and discuss their basic properties. In Section 5 we introduce the double mapping cylinder. We construct models in $\mathcal{R}$ of the real projective plane and 3-dimensional real projective space. Furthermore, we obtain some results on maps of (co-) triads in $\mathcal{R}$, related to those by J.P. May in [12]. By way of application we describe a Hopf construction in Section 6, obtaining a model (in $\mathcal{R}$) of a generator of the homotopy group $\pi_3(S^2)$.

2. Relational structures

Definition 2.1. A *reflexive binary relational structure* $\mathbf{X} = (X, \theta)$ (for brevity, simply *relational structure*) is a set X together with a reflexive relation

$$\theta \subseteq X \times X.$$

A morphism of relational structures $f : \mathbf{X} \to \mathbf{Y}$ is a function $f : X \to Y$ satisfying the condition that $(x_1, x_2) \in \theta_X \Rightarrow (fx_1, fx_2) \in \theta_Y$. Thus we obtain a category $\mathcal{R}$ (in [1] $\mathcal{R}$ is called **Rere**). The statement $(x_1, x_2) \in \theta_X$ will sometimes be expressed in the form $x_1 \to x_2$. The forgetful functor from $\mathcal{R}$ to **Set** will be denoted by U, and we note that there is another functor V from $\mathcal{R}$ to **Set** (the functor selects the set of relations).

The relation θ will in general be neither symmetric nor antisymmetric nor transitive, but of course the full subcategory **Poset** of all objects (X, θ) for which the relation θ is a partial order, is of special significance. Relational structures have been studied by various authors, for instance, in the book [1] there is frequent reference to this category. See also [15] and [10]. Further examples of relational structures which are not posets are given in Section 5.

Remark 2.2. We find it convenient to deal rather informally with other categories of which the objects are diagrams in $\mathcal{R}$. For simplicity and when there is no ambiguity, we shall refer to a relational structure (X, θ) without explicitly specifying the relation θ.

As mentioned previously, Larose and Tardif [10] show how to define homotopy groups in $\mathcal{R}$ in terms of morphisms in $\mathcal{R}$, and they prove, for finite pointed posets $(X, *)$, that these groups $\sigma_k(X, *)$ are naturally isomorphic to the homotopy groups $\pi_k(|\mathcal{K}X|, *)$. The corresponding notion of relative homotopy groups for relational structures is introduced in [18], where also it is shown how to obtain the exact homotopy sequence for a pair. Thus we may describe a morphism of $\mathcal{R}$ (or a morphism of pairs of objects of $\mathcal{R}$) as a *weak equivalence* if the usual conditions on homotopy groups are satisfied by the relevant σ-homotopy groups.

Subobjects 2.3. The category $\mathcal{R}$ is a *construct*, i.e. its objects are structured sets and the morphisms are structure preserving functions (see [1, 5.1 on p. 53 and 3.3 on p. 14]). Given an object (X, θ_X) of $\mathcal{R}$, by a *regular subobject* of (X, θ_X) we mean an object (Y, θ_Y) of $\mathcal{R}$ such that $Y \subset X$ and such that $(y_1, y_2) \in \theta_Y$ if and only if $(y_1, y_2) \in \theta_X$. In particular the inclusion of a regular subobject is a regular monomorphism in $\mathcal{R}$, i.e. the equalizer of a parallel pair of morphisms of $\mathcal{R}$.

Barycentric subdivision 2.4. We recall from [10] that the barycentric subdivision X' of an object X in $\mathcal{R}$ is defined to be the poset (under set inclusion) of all finite *chains* in X. Here we understand a chain in X to be the image in X of a morphism $f : C \to X$ where C is a totally ordered finite set. Barycentric subdivision can be seen to be a functor from $\mathcal{R}$ to **Poset**. We further note that in [10], for any $\mathcal{R}$-object X, a $\mathcal{R}$-morphism $p : X' \to X$ is shown to exist, such that for every chain τ in X, $p(\tau)$ is an upper bound of τ. The map p induces an isomorphism of $(\sigma-)$ homotopy groups and is unique if X is antisymmetric. We refer to any such map as a *barycentric retraction* noting however that although a retraction as a function between sets, the map p is not a retraction in $\mathcal{R}$.

In Chapter 1 we refered to the geometric realization of a poset P as being obtained by passing through the simplicial complex obtained from P. We give a direct description below:

Geometric realization 2.5. Geometric realization of a poset P (see [10]). For any function $h : P \to [0, 1]$ the *support* $\mathrm{st}(h)$ of h is the subset of P defined by

$\mathrm{st}(h) = \{p \in P : h(p) \neq 0\}$.

Now consider the set:

$|P| = \{h \in [0, 1]^P : \mathrm{st}(h)$ is a finite chain in P and $\Sigma_{p \in P}\, h(p) = 1\}$.

The set $|P|$ is equipped with the *coherent topology* (see [16, p. 5]) and the resulting topological space is called the *geometric realization* of P. Geometric realization is a functor **Poset** $\to$ **Top**, but is not a full functor.

Remark 2.6. One difficulty we encounter when using barycentric subdivision and retractions is that for a morphism $g : X \to Y$, a square of the form

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "\alpha"'] & Y' \arrow[d, "\beta"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

is not necessarily commutative. However we do have the following:
If g is injective, then for any choice of α we can find a certain β such that the square above is commutative.

$\mathcal{R}$-model 2.7. An $\mathcal{R}$-morphism $f : X_0 \to X_1$ is a *model* for the **Top**-morphism $g : A_0 \to A_1$ if there are weak equivalences h_0 and h_1 making the following square commutative.

$$\begin{tikzcd} |X_0'| \arrow[r, "|f'|"] \arrow[d, "h_0"'] & |X_1'| \arrow[d, "h_1"] \\ A_0 \arrow[r, "g"'] & A_1 \end{tikzcd}$$

Examples of finite **Poset**-models of Hopf maps appear in [8] and [7]. A related example appears in [9].

Product 2.8. We recall that the *product* of relational structures X and Y is the structure $X \times Y$ on the product of their underlying sets, where $(x, y) \to (x', y')$ if and only if $x \to x'$ and $y \to y'$.

Fences 2.9. The infinite *fence* F is the poset below.

$$0 \to 1 \leftarrow 2 \to 3 \leftarrow 4 \to 5 \ldots .$$

Thus F is the poset of which the underlying set is the set $\mathbb{N}$ of all non-negative integers, and the partial order $\theta \subset \mathbb{N} \times \mathbb{N}$ is defined as follows.

$$(k, l) \in \theta \quad \Leftrightarrow \quad l = k \text{ or } |k - l| = 1 \text{ and } k \text{ is even.}$$

For any $k \in \mathbb{N}$, F_k denotes the subobject $\{0, 1, 2, \ldots, k\}$ of F.

3. Pushouts

As mentioned before, a particularly convenient aspect of $\mathcal{R}$ is the simplicity of the construction of pushouts. We shall often refer to the following sketch which represents a commutative diagram in $\mathcal{R}$.

$$\begin{tikzcd} (X_0, \theta_0) \arrow[r, "f_2"] \arrow[d, "f_1"'] & (X_2, \theta_2) \arrow[d, "g_2"] \\ (X_1, \theta_1) \arrow[r, "g_1"'] & (X_3, \theta_3) \end{tikzcd} \tag{3A}$$

Since f_2 is a morphism, $(x_0, x_0') \in \theta_0 \Rightarrow (f_2(x_0), f_2(x_0')) \in \theta_2$ and hence the function $f_2 \times f_2 : X_0 \times X_0 \to X_2 \times X_2$ restricts to a function $Vf_2 : \theta_0 \to \theta_2$. In this way (3A) induces the following pair of squares in the category of sets (and here U and V are the **Set**-valued functors defined in 2.1).

$$\begin{tikzcd} X_0 \arrow[r, "Uf_2"] \arrow[d, "Uf_1"'] & X_2 \arrow[d, "Ug_2"] \\ X_1 \arrow[r, "Ug_1"'] & X_3 \end{tikzcd} \qquad \begin{tikzcd} \theta_0 \arrow[r, "Vf_2"] \arrow[d, "Vf_1"'] & \theta_2 \arrow[d, "Vg_2"] \\ \theta_1 \arrow[r, "Vg_1"'] & \theta_3 \end{tikzcd} \tag{3B}$$

Proposition 3.1. *Consider the commutative square in $\mathcal{R}$ given in (3A). The following two conditions are equivalent.*
(a) *Diagram (3A) is a pushout square.*
(b) *In the category of sets, the squares in (3B) are both pushouts.*

Proof. Let us assume that (a) holds and suppose we have the following commutative diagram in $\mathcal{R}$

$$\begin{tikzcd}
(X_0,\theta_0) \arrow[r] \arrow[d] & (X_2,\theta_2) \arrow[d] \arrow[ddr, "\phi_2"] & \\
(X_1,\theta_1) \arrow[r] \arrow[drr, "\phi_1"'] & (X_3,\theta_3) & \\
& & (Y,\theta_Y)
\end{tikzcd}, \tag{3C}$$

where we restrict θ_Y to being of the form $\theta_Y = Y \times Y$ (so that each function ϕ_1, ϕ_2 into Y is a $\mathcal{R}$-morphism). Then since diagram (3A) is a pushout square, it follows that the first square of (3B) is a pushout in Sets. Now let us choose X and θ_X to be the pushouts of the cotriads

$$X_1 \xleftarrow{Uf_1} X_0 \xrightarrow{Uf_2} X_2 \qquad \theta_1 \xleftarrow{Vf_1} \theta_0 \xrightarrow{Vf_2} \theta_2.$$

Since the square (3A) is commutative, it follows that $X = X_3$ and $\theta_X \subseteq \theta_3$. If in diagram (3C) we take $(Y, \theta_Y) = (X, \theta_X)$, then since diagram (3A) is a pushout we have $\theta_3 \subseteq \theta_X$. Thus we have shown that (a) implies (b).

Conversely, let us assume (b) and suppose that the diagram (3C) is given. Then using the pushout status of the square on the left hand side of diagram (3B) we can find a set-theoretic function $\phi : X_3 \to Y$ such that $\phi \circ Ug_1 = \phi_1$ and $\phi \circ Ug_2 = \phi_2$. Since the square on the right hand side of diagram (3B) is a pushout, it follows that ϕ is a $\mathcal{R}$-morphism. Thus the statement (a) follows from (b). □

Definition 3.2. Consider an object (X, θ_X) in $\mathcal{R}$. Let (X_1, θ_1) and (X_2, θ_2) be objects such that $X_1 \cup X_2 = X$, $X_0 = X_1 \cap X_2$ and $\theta_0 = \theta_1 \cap \theta_2$. We say that the cotriad $(X_1,\theta_1) \xleftarrow{f_1} (X_0,\theta_0) \xrightarrow{f_2} (X_2,\theta_2)$ is *conservative* if f_1 and f_2 are regular monomorphisms. Furthermore, we say that the square of regular monomorphisms (and we introduce a new notation for squares)

$$[X_0, X_1, X_2, X] = \begin{tikzcd}
(X_0,\theta_0) \arrow[r] \arrow[d] & (X_1,\theta_1) \arrow[d] \\
(X_2,\theta_2) \arrow[r] & (X,\theta_X)
\end{tikzcd}$$

is a *conservative square* and the triad $(X_1, \theta_1) \to (X, \theta) \leftarrow (X_2, \theta_2)$ a *conservative triad* if given any $a, b \in X$ such that $(a, b) \in \theta_X$, then $(a, b) \in \theta_1$ or $(a, b) \in \theta_2$.

We shall see that conservative squares have useful properties. Example 3.5 in this section shows why they are appropriate.

Proposition 3.3. *Suppose that in the commutative diagram* (3A) *every morphism is a regular monomorphism. Then the square is a pushout square if and only if it is a conservative square.*

Proof. Suppose that $[X_i]$ is a pushout square in $\mathcal{R}$. By Proposition 3.1, the square $[\theta_i]$ is a pushout square. If $(x, x') \in \theta_3$, then either $x_1, x_2 \in \operatorname{Im} g_1$ and $(x, x') \in \theta_1$ or $x, x' \in \operatorname{Im} g_2$ and $(x, x') \in \theta_2$. Thus $[X_i]$ is a conservative square.
Conversely, suppose that $[X_i]$ is a conservative square. Now consider any pair of morphisms $\alpha_1 : X_1 \to Y$ and $\alpha_2 : X_2 \to Y$ in $\mathcal{R}$ for which the following identity holds:

$$f_1 \circ \alpha_1 = f_2 \circ \alpha_2.$$

Then since we have a pushout square in **Set**, it follows that there is a unique set-theoretic function $\phi : X \to Y$ satisfying the condition:

$$\alpha_1 = \phi \circ g_1 \quad \text{and} \quad \alpha_2 = \phi \circ g_2.$$

We need only prove that ϕ is an $\mathcal{R}$-morphism. Now suppose that $(x, x') \in \theta_X$ and $x \neq x'$. Since $[X_i]$ is conservative we have

$$x, x' \in \operatorname{Im} g_i, \quad \text{where } i = 1 \text{ or } i = 2.$$

Since g_i is a regular monomorphism, $g_i^{-1}(x) \neq g_i^{-1}(x')$ and $g_i^{-1}(x) \to g_i^{-1}(x')$. Since α_i is an $\mathcal{R}$-morphism, we have $\alpha_i g_i^{-1}(x) \to \alpha_i g_i^{-1}(x')$, i.e., $\phi(x) \to \phi(x')$. Thus ϕ is an $\mathcal{R}$-morphism and we have shown that $[X_i]$ is a pushout square. □

Proposition 3.4. *Suppose that* $(X; X_1, X_2)$ *is a conservative triad in* $\mathcal{R}$. *Then*
(a) $X_1' \cap X_2' = (X_1 \cap X_2)'$,
(b) $X' = X_1' \cup X_2'$,
(c) *The triad* $(X'; X_1', X_2')$ *is conservative.*

Proof. The statement (a) is easily seen to be true (and for this we do not require $(X; X_1, X_2)$ to be conservative).
(b) Consider any $\tau = \{x_1, x_2, \ldots, x_k\} \in X'$. We can assume that for any $i, j \in \mathbb{N}$ with $i \leq j \leq k$, we have $(x_i, x_j) \in \theta$. Now if $\tau \notin X_1' \cup X_2'$, then there exist $i, j \in \{1, 2, \ldots, k\}$ such that $x_i \in X_1 \backslash X_2$ and $x_j \in X_2 \backslash X_1$, but this is not possible when dealing with a conservative triad. Thus we must have $\tau \in X_1' \cup X_2'$, and so (b) holds.
(c) This follows easily. □

Our preference for conservative squares and regular subobjects is motivated by the following example in which we form the pushout of a cotriad consisting of a pair of monomorphisms, but with one of them not being regular.

Example 3.5. Let $X_0 = \{a, b\}$ be an antichain, i.e., $\theta_0 = \{(a, a), (b, b)\}$. Let $X_1 = \{a, z, b\}$ with $\theta_1 = \{(a, a), (a, z), (z, z), (z, b), (b, b)\}$ and let $X_2 = \{x, y\}$ with $\theta_{X_2} = \{(x, x), (y, y), (x, y), (y, x)\}$. Let f_1 be the inclusion (which is regular) and let $f_2 : X_0 \to X_2$ be the morphism such that $f_2(a) = x$ and $f_2(b) = y$. Then the pushout of the cotriad

$$X_1 \xleftarrow{f_1} X_0 \xrightarrow{f_2} X_2 \tag{3D}$$

yields an object $Y =$
$$\begin{tikzcd} & x \arrow[dl] \arrow[dd, shift right] \\ z \arrow[dr] & \\ & y \arrow[uu, shift right] \end{tikzcd}$$
which is weakly contractible in $\mathcal{R}$.

On the other hand, if we pick a topological model of the cotriad (3D), with f_1 being modeled by a cofibration then the pushout (in **Top**) would be of the weak homotopy type of a circle.

4. Homotopy pushouts

The adjunction type construction in [3] of Dold and Lashof proved very versatile, and the basic requirements for constructing maps in this way was discussed in [6]. The interplay between pullback and pushout when studying homotopy-theoretic fibres are presented in the paper [11] of Mather. We address the analogous phenomena in $\mathcal{R}$.

Notation 4.1. Consider a commutative diagram in $\mathcal{R}$ such as below.

$$\begin{tikzcd}[row sep=small, column sep=small] & X_0 \arrow[rr] \arrow[dl] \arrow[dd] & & X_2 \arrow[dl] \arrow[dd] \\ X_1 \arrow[rr, crossing over] \arrow[dd] & & X_3 & \\ & Y_0 \arrow[rr] \arrow[dl] & & Y_2 \arrow[dl] \\ Y_1 \arrow[rr] & & Y_3 \arrow[from=uu, crossing over] & \end{tikzcd} \tag{4A}$$

The diagram can be considered as consisting of a square $[X_i]$ (the top of the "box") and a square $[Y_i]$, as well as a morphism $[\varphi_i]$ from $[X_i]$ to $[Y_i]$. If each of the morphisms $\varphi_i : X_i \to Y_i$ are weak equivalences in $\mathcal{R}$, then we shall refer to $[\varphi_i]$ as being a weak equivalence of squares.

Definition 4.2. (a) Suppose that the objects of diagram (3A) are pointed posets. Diagram (3A) is said to be a *homotopy pushout square* (and we abbreviate to HPS) if the corresponding square of geometric realisations is a homotopy pushout square in the category of pointed topological spaces.

(b) More generally in $\mathcal{R}$, diagram (3A) is said to be an HPS if for some $k \in \mathbb{N}$ there exist a sequence of weak equivalences of squares as follows,

$$\begin{tikzcd} {[X_i^{(0)}]} \arrow[d, "\sim"'] & {[X_i^{(2)}]} \arrow[dl, "\sim"'] \arrow[d, "\sim"'] & {[X_i^{(4)}]} \arrow[dl, "\sim"'] & \cdots & {[X_i^{(2k)}]} \arrow[dl, "\sim"'] \\ {[X_i^{(1)}]} & {[X_i^{(3)}]} & \cdots & {[X_i^{(2k-1)}]} & \end{tikzcd},$$

where the object $[X_i^{(0)}]$ is the diagram (3A), every object of the square $[X_i^{(2k)}]$ is a poset and $[X_i^{(2k)}]$ is an HPS (in the sense of (a) above).

Terminology 4.3. A square in $\mathcal{R}$ is a called a PHP square if it is both a pushout square and a homotopy pushout square.

Proposition 4.4. *In diagram (3A) suppose that all the arrows are regular monos, with $X_0 = X_1 \cap X_2$, and that the square is conservative. Then the diagram is a* PHP *square.*

Proof. By Proposition 3.4(c) the triad $(X_3'; X_1', X_2')$ is conservative. By 3.4(a) we have $X_1' \cap X_2' = (X_1 \cap X_2)'$. Let us write $X_1 \cap X_2 = X_0$. Choosing a barycentric retraction $r_3 : X_3' \to X_3$, there will be induced barycentric retractions $r_i : X_i' \to X_i$ for each $i \in \{0, 1, 2\}$. By [10] the maps $r_i : X_i' \to X_i$ are weak equivalences. Moreover the objects X_i' are all posets. □

5. The double mapping cylinder in $\mathcal{R}$

Given any $\mathcal{R}$-morphism $g : A \to B$, the *mapping F_1-cylinder* Z_g of g is the object obtained as the pushout:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "a_1"'] & B \arrow[d] \\ A \times F_1 \arrow[r, "g'"'] & Z_g \end{tikzcd},$$

where $a_1 : a \mapsto (a, 1)$ and F_1 is the fence $0 \to 1$. We note that the map $a_0 : a \mapsto (a, 0)$ of A into $A \times F_1$ gives rise to a map $\overline{g} = g' \circ a_0 : A \to Z_g$. Given a cotriad (h, g) as below,

$$C \xleftarrow{h} A \xrightarrow{g} B,$$

the *double mapping F_2-cylinder* $Z(h, g)$ is defined to be the object obtained via the pushout square:

$$\begin{tikzcd} A \arrow[r, "\overline{g}"] \arrow[d, "\overline{h}"'] & Z_g \arrow[d] \\ Z_h \arrow[r] & Z(h, g) \end{tikzcd} \tag{5A}$$

and we note that the square is conservative.

Alternatively the double mapping cylinder can be described as follows. Let $\mathbb{I} = \{-1, 0, 1\}$ be the poset in which the partial order is given by $0 \to -1,\ \ 0 \to 1$. For the following cotriad in $\mathcal{R}$,

$$(X_1, \theta_1) \xleftarrow{f_1} (X_0, \theta_0) \xrightarrow{f_2} (X_2, \theta_2), \tag{5B}$$

the *double mapping cylinder* $Z(f_1, f_2) = (Z, \theta_Z)$, where Z is is obtained from $X_1 + (X_0 \times \mathbb{I}) + X_2$ (a disjoint sum) by making the identifications:
for each $x \in X_0$, $(x, -1) \sim f_1(x)$ *and* $(x, 1) \sim f_2(x)$
and choosing θ_Z to be the smallest set of relations on Z such that the obvious functions from $X_1,\ X_0 \times \mathbb{I}$ and X_2 into Z are $\mathcal{R}$-morphisms.
For each $i = 1, 2$ there are weak equivalences $\zeta_i : Z_{f_i} \to X_i$ making the following diagram commutative.

$$\begin{tikzcd} Z_{f_1} \arrow[d, "\zeta_1"'] & (X_0, \theta_0) \arrow[l] \arrow[r] \arrow[d, "1"] & Z_{f_2} \arrow[d, "\zeta_2"] \\ (X_1, \theta_1) & (X_0, \theta_0) \arrow[l, "f_1"'] \arrow[r, "f_2"] & (X_2, \theta_2). \end{tikzcd} \tag{5C}$$

Thus there is a natural map $\zeta : Z(f_1, f_2) \to (X, \theta_X)$, where (X, θ_X) is the pushout of the cotriad (5B).

We require the following terminology, the equivalent of adjunction space in topology.

Adjunction object 5.1. Consider an object X_1, a subobject X_0 of X_1 and a morphism $f : X_0 \to X_2$. Then the object obtained as the pushout of the cotriad formed by f and the inclusion map will be denoted by $X_1 \cup_f X_2$.

Proposition 5.2. *If the square* (3A) *is a pushout square, then it is a homotopy pushout square if and only if the natural map* $\zeta : Z(f_1, f_2) \to X_3$ *is a weak equivalence.*

Proof. We have noted that the square (5A) is conservative. In particular the cotriad $M_h \xleftarrow{\overline{h}} A \xrightarrow{\overline{g}} M_g$ is conservative. Therefore via Proposition 4.4 we can deduce that its pushout square is an HPS. The rest is clear from the very definition of homotopy pushout square. □

The following definition is quoted from [18] and is based on a similar one in topology, which can be found in the paper [12] of J.P. May.

Definition 5.3 (*From [18]*). A map $p : (X, A) \to (Y, B)$ of pairs in $\mathcal{R}$ is a 0-*equivalence* if the first condition below holds. If n is a positive integer then f is said to be an n-*equivalence* if both conditions hold.

(1) $\mathrm{Im}[\sigma_0(A) \to \sigma_0(X)] = p_*^{-1}\mathrm{Im}[\sigma_0(B) \to \sigma_0(Y)]$,

(2) For every $a \in A$, and with $b = p(a)$, the function $p_* : \sigma_r(X, A, a) \to \sigma_r(Y, B, b)$ is bijective whenever $r < n$ and surjective for $r = n$.

The map p is said to be a *weak equivalence* if it is an n-equivalence for all $n > 0$.
A map $p : X \to Y$ is said to be a *quasifibration* if for every $y \in Y$ and $G = p^{-1}(y)$, the induced map of pairs $(X, G) \to (Y, y)$ is a weak equivalence.

Theorem 5.4. *Suppose that diagram (5D) is commutative and let W and Z be the double mapping cylinders of the cotriads in the top row and the bottom row respectively. Let $k \in \mathbb{N}$. Suppose that for each $i \in \{1, 2\}$ the map $(W_i, W_0) \to (Z_i, Z_0)$ is a k-equivalence.*

$$\begin{tikzcd} W_1 \arrow[d, "\zeta_1"'] & W_0 \arrow[l] \arrow[r] \arrow[d, "\zeta_0"'] & W_2 \arrow[d, "\zeta_2"] \\ Z_1 & Z_0 \arrow[l] \arrow[r] & Z_2. \end{tikzcd} \tag{5D}$$

Then $(W, W_0) \to (Z, Z_0)$ is a $(k + 1)$-equivalence and for each $i \in \{1, 2\}$ the map $(W, W_i) \to (Z, Z_i)$ is a k-equivalence.

Proof. We consider the following maps of squares:

$[W_0', \ldots, W'] \to [W_0, \ldots, W]$ is a weak equivalence,

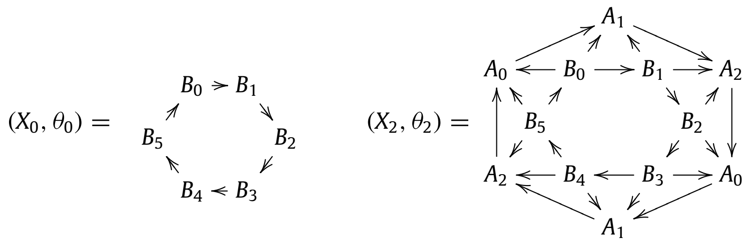
$[Z_0', \ldots, Z'] \to [Z_0, \ldots, Z]$ is a weak equivalence.

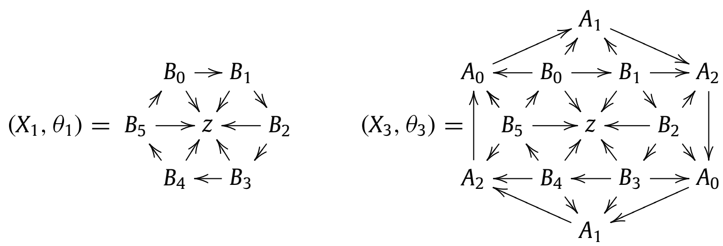
On the level of the barycentric subdivisions which are posets, we can apply topological results. For each $i \in \{1, 2\}$ the map $(W_i', W_0') \to (Z_i', Z_0')$ is a k-equivalence. The assertion of Theorem 5.4 now follows by [17, Theorem 0.4]. □

Theorem 5.5. *Suppose that $f_1 : X_0 \to X_1$ and $f_2 : X_0 \to X_2$ are $\mathcal{R}$-models of a pair of* **Top**-*morphisms $g_1 : A_0 \to A_1$ and $g_2 : A_0 \to A_2$, and let $M(g_1, g_2)$ be the relevant double mapping cylinder. Then $Z(f_1, f_2)$ is an $\mathcal{R}$-model of $M(g_1, g_2)$.*

Proof. By design, $Z(f_1', f_2')$ is a model of $M(|f_1'|, |f_2'|)$. By Theorem 5.4 it follows that there is a weak equivalence $Z(f_1', f_2') \to Z(f_1, f_2)$. The rest of the proof is simple. □

Example 5.6 (*Real Projective Plane*). We consider the case of diagram (3A) in which the objects are as indicated below. (In a given object, points with the same label are understood to be identified.)





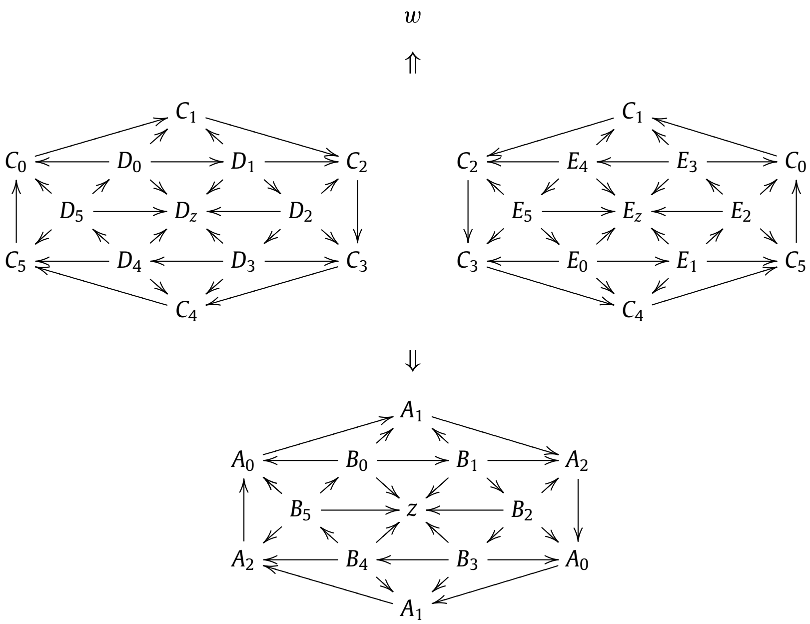
Note that the square of maps is conservative, and hence PHP. It follows from Theorem 5.5 that the object (X_3, θ_3) is a model in $\mathcal{R}$ of the real projective plane.

Remark 5.7. The reader familiar with the usual construction of $\mathbb{R}P^2$ in **Top** *viz.* attaching a 2-disk to a circle by means of a degree two map $S^1 \to S^1$ may wonder what has happened here. We observe that a degree two map

$$\begin{tikzcd} & B_0 \arrow[r] & B_1 \arrow[dr] & & & & A_0 \arrow[dr] & \\ B_5 \arrow[ur] & & & B_2 \arrow[dl] & \longrightarrow & A_2 \arrow[ur] & & A_1 \arrow[ll] \\ & B_4 \arrow[ul] & B_3 \arrow[l] & & & & & \end{tikzcd}$$

such that $(B_0, B_3) \mapsto A_0$, $(B_1, B_4) \mapsto A_1$, $(B_2, B_5) \mapsto A_2$ has been replaced by the inclusion of the domain into an F_1-mapping cylinder. Indeed the object constructed is a union (with common domain) of a pair of F_1-mapping cylinders to form an $\mathbb{I}$-double mapping cylinder.

Example 5.8 (*Real Projective 3-space*).



The diagram above has three 'levels'. Level one consists of the single point w. The (double-shafted) Up-arrow is intended to suggest a sheaf of individual relational arrows to w from each point in level 2 consisting of the C_i, D_i and E_i, D_z and E_z. The points and arrows of level 2 constitute a 20-point model (X_0, θ_0) of a 2-sphere with equatorial points C_0 through C_5.

The object in level 3 consists of the model of $\mathbb{R}P^2$ obtained from (X_0, θ_0) on factoring out by the antipodal pair relation:

$$\{C_0, C_3\} = A_0, \qquad \{C_1, C_4\} = A_1, \qquad \{C_2, C_5\} = A_2,$$
$$\{D_i, E_i\} = B_i \quad (0 \leq i \leq 5), \qquad \{D_z, E_z\} = z.$$

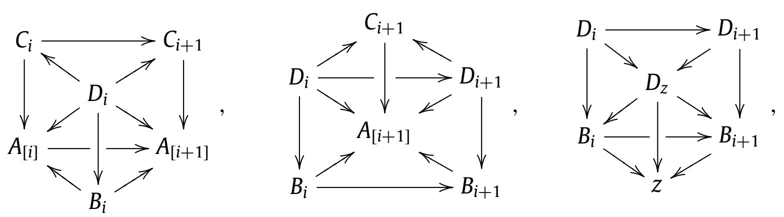
The Down-arrow from level 2 to level 3 represents relational arrows as described below. The diagram as a whole describes a 31-point $\mathcal{R}$ model of $\mathbb{R}P^3$. To understand this we need to recognise the model constructed above as the pushout object (X_3, θ_3) of the conservative cotriad

$$(X_1, \theta_1) \xleftarrow{f_1} (X_0, \theta_0) \xrightarrow{f_2} (X_2, \theta_2),$$

where (X_0, θ_0) is the central 2-sphere model in level 2, f_1 is its inclusion into the union of level one and level 2 and f_2 is its inclusion into the union of level 2 and level 3. To list the relational arrows from level 2 to level 3, we need to consider the $\mathcal{R}$-morphism g from (X_0, θ_0) to the model of $\mathbb{R}P^2$ in level 3 that identifies antipodal points. Specifically, we have

$$g(C_0) = g(C_3) = A_0, \qquad g(C_1) = g(C_4) = A_1, \qquad g(C_2) = g(C_5) = A_2,$$
$$g(D_i) = g(E_i) = B_i \quad (0 \leq i \leq 5), \qquad g(D_z) = g(E_z) = z.$$

The required relational arrows from level 2 to level 3 are precisely the arrows of the F_1-mapping cylinder of g. This object is hard to sketch in its entirety but it can be dissected into a number of triangular prisms. In the following, for an integer k in the subscript, we blur the distinction between k and its residue class modulo 6, while we denote by $[k]$ the residue of k modulo 3. Note that a diagonal arrow (of negative slope) is missing from the rear face of each prism. For each $(0 \leq i \leq 5)$ there are "prism" diagrams as follows.



$$
\begin{tikzcd}
C_i \arrow[rr] \arrow[dd] & & C_{i+1} \arrow[dd] \\
& E_{i+3} \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] \arrow[dd] & \\
A_{[i]} \arrow[rr] & & A_{[i+1]} \\
& B_{i+3} \arrow[ul] \arrow[ur] &
\end{tikzcd}
,\qquad
\begin{tikzcd}
& C_{i+4} \arrow[dd] & \\
E_i \arrow[ur] \arrow[rr] \arrow[dr] \arrow[dd] & & E_{i+1} \arrow[ul] \arrow[dl] \arrow[dd] \\
& A_{[i+4]} & \\
B_i \arrow[ur] \arrow[rr] & & B_{i+1} \arrow[ul]
\end{tikzcd}
,\qquad
\begin{tikzcd}
E_i \arrow[rr] \arrow[dr] \arrow[dd] & & E_{i+1} \arrow[dl] \arrow[dd] \\
& E_z \arrow[dl] \arrow[dr] \arrow[dd] & \\
B_i \arrow[rr] \arrow[dr] & & B_{i+1} \arrow[dl] \\
& z &
\end{tikzcd}
.
$$

6. Join and Hopf construction

The object of this section is to interpret Hopf's construction in the category $\mathcal{R}$. In the literature there are many descriptions of the construction (for topological spaces, see for example [5], pp. 334–335).

In [8] the *non-Hausdorff join* of a pair of posets is defined, as well as *non-Hausdorff cones* and *suspensions*. These concepts have analogues in $\mathcal{R}$.

Definition 6.1 (*cf [8]*). We define the *join* $(X, \theta_X) \star (Y, \theta_Y)$ of the relational structures (X, θ_X) and (Y, θ_Y), to be the structure

$$(X + X \times Y + Y,\ \theta_\star),$$

where $\theta_\star$ is the smallest relation on $X + X \times Y + Y$ containing the existing relations in (X, θ_X), (Y, θ_Y) and in the product $(X, \theta_X) \times (Y, \theta_Y)$ together with all the relations of the following type (here we consider $x, x_1 \in X$ and $y, y_1 \in Y$):

$(x, y) \to x_1$, whenever $(x, x_1) \in \theta_X$,

$(x, y) \to y_1$, whenever $(y, y_1) \in \theta_Y$.

It can be verified that $(X, \theta_X) \star (Y, \theta_Y)$ coincides with the double mapping fence-cylinder $Z(\pi_1, \pi_2)$, where π_1 and π_2 refer to the projections from $(X, \theta_X) \times (Y, \theta_Y)$ on to the factor structures.

Suspension 6.2. If (X, θ_X) is a relational structure, the *suspension* $\mathbb{S}(X, \theta_X)$ can be regarded as the double mapping cylinder $Z(f_1, f_2)$, where f_1 and f_2 are projections onto singletons as shown in

$$s \xleftarrow{f_1} (X, \theta_X) \xrightarrow{f_2} n\cdot$$

It is easy to see that there is an antisymmetric n-point model of the circle in $\mathcal{R}$ for $n \geq 3$. The example below on the left presents the case $n = 4$. The object on the right hand side is a symmetric object.

Example 6.3.

$$
D_4 = \begin{tikzcd}
& 1 \arrow[dl] & \\
2 \arrow[dr] & & 0 \arrow[ul] \\
& 3 \arrow[ur] &
\end{tikzcd}
;\qquad
\begin{tikzcd}
& 1 \arrow[dl, leftrightarrow] & \\
2 \arrow[dr, leftrightarrow] & & 0 \arrow[ul, leftrightarrow] \\
& 3 \arrow[ur, leftrightarrow] &
\end{tikzcd}
= \overline{D}_4. \tag{6A}
$$

In the case of D_4 we understand that

$$\theta = \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 1), (1, 2), (2, 3), (3, 0)\},$$

and for $\overline{D}_4$ we augment the latter relation to form its *symmetrization*. To see that the objects are models of the circle, note that in both cases the barycentric subdivisions turn out to be the poset

$$
(D_4)' = (\overline{D}_4)' = \begin{tikzcd}
12 & 1 \arrow[l] \arrow[r] & 01 \\
2 \arrow[u] \arrow[d] & & 0 \arrow[u] \arrow[d] \\
23 & 3 \arrow[l] \arrow[r] & 30
\end{tikzcd}
. \tag{6B}
$$

The topological circle is, of course, the underlying space of a topological group. However, models of the circle in other categories do not always have an appropriate multiplication. For example, it was pointed out in [8] that the (4-point crown), i.e. the poset

$$C_4 = \{1,\ -1,\ i,\ -i \mid 1 \to i,\ 1 \to -i,\ -1 \to i,\ -1 \to -i\},$$

known to be a model of the circle, does not support the complex number multiplication since that multiplication fails to be order-preserving. An order-preserving function $m : C_8 \times C_8 \to C_4$ was eventually found in [8] but the point 1 failed to be a strict unit for m, merely a *homotopy unit*, in the sense that the *axial* maps $x \mapsto m(1, x)$ and $x \mapsto m(x, 1)$ were shown to be weak homotopy equivalences. Such constructions are easier to achieve in $\mathcal{R}$.

Example 6.4. Although the following table does not describe an $\mathcal{R}$-morphism $D_4 \times D_4 \to D_4$ with 0 as identity element

$$
\begin{array}{c|cccc}
\times & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 2 & 3 & 0 \\
2 & 2 & 3 & 0 & 1 \\
3 & 3 & 0 & 1 & 2
\end{array}
\tag{6C}
$$

(there is a failure of preservation along diagonals) the table (6C) succeeds in defining a multiplication in $\mathcal{R}$ of form: $\mathrm{op}(D_4) \times D_4 \to \overline{D}_4$ with 0 as strict unit. (This time the diagonal arrows in the product run between points sent to equal integers.)

Example 6.5. The suspension of $\overline{D}_4$ is the octahedral structure:

$$
\begin{tikzcd}
& n & & & \\
0 \arrow[ur] \arrow[r, leftrightarrow] \arrow[dr] & 1 \arrow[u] \arrow[d] \arrow[r, leftrightarrow] & 2 \arrow[ul] \arrow[dl] \arrow[r, leftrightarrow] & 3 \arrow[ull] \arrow[dll] \arrow[r, leftrightarrow] & {} \\
& s & & &
\end{tikzcd}
\tag{6D}
$$

Example 6.6. If $m : A \times B \to Y$ is a $\mathcal{R}$-morphism, then its *Hopf construction* is the morphism $A * B \to SY$ induced by functoriality.

The object $\mathrm{op}(D_4) \star D_4$ is indicated in the sketch (6E). Note that the arrows π_1 and π_2 must be interpreted in the sense of the relations specified in (6.1).

$$
\begin{tikzcd}
& & 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & {} \\
& & & \arrow[r, phantom, "\Uparrow \pi_2"] & & & \\
0' & & (0',0) \arrow[r] & (0',1) \arrow[r] & (0',2) \arrow[r] & (0',3) \arrow[r] & {} \\
1' \arrow[u] & & (1',0) \arrow[u] \arrow[ur] \arrow[r] & (1',1) \arrow[u] \arrow[ur] \arrow[r] & (1',2) \arrow[u] \arrow[ur] \arrow[r] & (1',3) \arrow[u] \arrow[ur] \arrow[r] & {} \\
2' \arrow[u] & \overset{\pi_1}{\Leftarrow} & (2',0) \arrow[u] \arrow[ur] \arrow[r] & (2',1) \arrow[u] \arrow[ur] \arrow[r] & (2',2) \arrow[u] \arrow[ur] \arrow[r] & (2',3) \arrow[u] \arrow[ur] \arrow[r] & {} \\
3' \arrow[u] & & (3',0) \arrow[u] \arrow[ur] \arrow[r] & (3',1) \arrow[u] \arrow[ur] \arrow[r] & (3',2) \arrow[u] \arrow[ur] \arrow[r] & (3',3) \arrow[u] \arrow[ur] \arrow[r] & {} \\
{} \arrow[u] & & {} \arrow[u] \arrow[ur] & {} \arrow[u] \arrow[ur] & {} \arrow[u] \arrow[ur] & {} \arrow[u] \arrow[ur] &
\end{tikzcd}
\tag{6E}
$$

We now display the $\mathcal{R}$ model of the Hopf construction of the multiplication (2.2), replacing the points of the diagram in (6E) by their images under the associated function:

$$
\begin{tikzcd}
& n \arrow[r] & n \arrow[r] & n \arrow[r] & n \arrow[r] & {} \\
s & 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & {} \\
s \arrow[u] & 1 \arrow[u] \arrow[ur] \arrow[r] & 2 \arrow[u] \arrow[ur] \arrow[r] & 3 \arrow[u] \arrow[ur] \arrow[r] & 0 \arrow[u] \arrow[ur] \arrow[r] & {} \\
s \arrow[u] & 2 \arrow[u] \arrow[ur] \arrow[r] & 3 \arrow[u] \arrow[ur] \arrow[r] & 0 \arrow[u] \arrow[ur] \arrow[r] & 1 \arrow[u] \arrow[ur] \arrow[r] & {} \\
s \arrow[u] & 3 \arrow[u] \arrow[ur] \arrow[r] & 0 \arrow[u] \arrow[ur] \arrow[r] & 1 \arrow[u] \arrow[ur] \arrow[r] & 2 \arrow[u] \arrow[ur] \arrow[r] & {} \\
{} \arrow[u] & {} \arrow[u] \arrow[ur] & {} \arrow[u] \arrow[ur] & {} \arrow[u] \arrow[ur] & {} \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

The result is a model of Hopf's map from S^3 to S^2. The reader will note that the model is somewhat simpler than the poset model given in [8].

Remark 6.7. We note that every fibre of the map h is isomorphic to D_4. In particular it is not hard to see that in fact h is a quasifibration, since every map $S^1 \to S^3$ is nullhomotopic.

Acknowledgment

The second-named author acknowledges a grant for research from the South African National Research Foundation.

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